Singularity Analysis of 3-DOF Planar Parallel Mechanisms via Screw Theory

Ilian A. Bonev
Dimiter Zlatanov
Clément M. Gosselin
Département de Génie Mécanique, Université Laval, Québec, QC, Canada G1K 7P4

1 Introduction
The direct kinematic problem has been studied in detail for all 3-DOF planar parallel mechanisms (PPMs) [1,2]. In a similar fashion, the singular configurations of all 3-DOF PPMs have been described in [3]. However, the authors have only presented a general approach for setting up the velocity equations and identifying all singular configurations, by dividing all PPMs into two classes. Exhaustive singularity analyses have been presented for only two 3-DOF PPMs [4,5].

This paper provides a comprehensive investigation of the singularities of PPMs. Such a detailed study yields valuable theoretical insight into the kinematics of PPMs. In practice, the presented results can be very useful in the selection of the optimal architecture for a given task. Furthermore, we point out several novel architectures with easy to determine singularities are identified.

2 Planar Instantaneous Kinematics via Screw Theory
The conventional process of deriving the input-output velocity equation for a parallel mechanism consists in differentiating the corresponding notation. Therefore, we can eliminate eight more pairs of symmetrical legs, where each pair leads to two kinematically equivalent PPMs. The elements of this basis are the three rotations about and three translations along the frame axes. The elements of $S$ can be also interpreted as wrenches, the basis vectors being the pure forces along and the moments about the coordinate axes. In fact, wrenches comprise the dual vector space $S^*$, i.e., they are linear forms defined on $S$. The action of a wrench on a twist is the instantaneous work contributed by the wrench during the motion along the twist and defines the reciprocal screw product of the underlying elements of $S$. When a wrench applies no power on a twist, their reciprocal product is zero, and it is said that the two are reciprocal.

The coordinates of a twist in the standard basis will be denoted by $\xi=(\omega,v) = (\omega_x,\omega_y,\omega_z,v_x,v_y,v_z)$. When an element of $S$ is seen as a wrench, its coordinates in the standard basis are given by $\xi=(f,m)=(f_x,f_y,f_z,m_x,m_y,m_z)$. As is customary in the literature, we will use the term screw to refer to a normalized* element of $S$, twist or wrench [12].

For planar mechanisms, all instantaneous motions (or twists) are, however, two subtler aspects of the method that are not easily to treat the twists (instantaneous motions) and wrenches (forces and moments) involved in the velocity and singularity analysis as $n$-dimensional so that the matrices involved are $n \times n$ [9,11]. Velocity analysis in such cases amounts to an $n$-dimensional version of screw calculus. However, screws and reciprocal screws (i.e., twists and wrenches) have different sets of $n$ coordinates. Unlike the general 6-DOF case, screws and reciprocal screws can no longer be thought of as elements of the same vector space. Three-dimensional planar screws will be the focus of Section 2.2.

Secondly, for the analysis to remain valid in all configurations, it is important to always find a maximal set of (independent) reciprocal screws. Otherwise, the method can misinterpret or even fail to detect certain singularities [9]. That issue will be discussed in Section 3.1.

2.2—Planar Twists and Wrenches. We assume that for all admissible reference frames the origin as well as the $x$ and $y$ axes are in the chosen plane of motion. For every given Cartesian frame in space we associate a standard basis, $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z,\tau,\pi,\iota\}$, in the six-dimensional space of twists, $S$. The elements of this basis are the three rotations about and three translations along the frame axes. The elements of $S$ can be also interpreted as wrenches, the basis vectors being the pure forces along and the moments about the coordinate axes. In fact, wrenches comprise the dual vector space $S^*$, i.e., they are linear forms defined on $S$. The action of a wrench on a twist is the instantaneous work contributed by the wrench during the motion along the twist and defines the reciprocal screw product of the underlying elements of $S$. When a wrench applies no power on a twist, their reciprocal product is zero, and it is said that the two are reciprocal.

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**Contributed by the Mechanisms and Robotics Committee for publication in the Journal of Mechanical Design. Manuscript received July 2001; revised October 2002. Associate Editor: D. C. H. Yang.**
are part of the three-dimensional screw system of instantaneous planar motion. The planar screw system is, in fact, \( \mathcal{E} = \text{Span}(\mathbf{q}, \mathbf{a}, \mathbf{r}) \). A planar twist will always have three of its coordinates, \( \alpha_z, \omega_z, \nu_z \), equal to zero. Hence, we will write the planar twists as three-dimensional vectors, \( \xi = (\omega, \nu_z) \).

The reciprocal screws used in singularity analysis are reciprocal to some but not all joint twists. Physically, they represent wrenches which, when applied to the output link can be resisted by (i.e., do no work on) the mechanism at certain conditions (when some active joints are locked), but not always. In other words, these screws never belong to the reciprocal system of the screw system of the mechanism twists. Indeed, screws reciprocal to all mechanism twists would be of no interest. For planar mechanisms, the reciprocal system, \( \mathcal{E}^* \), is identical to \( \mathcal{E} \) itself. Therefore, we can assume that the reciprocal screws are all in a 3-system complementary to \( \mathcal{E} \), for example, in \( \mathcal{W} = \text{Span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}) \). The elements of \( \mathcal{W} \) are pure forces in the \( xy \) plane as well as the pure moment about the \( z \) axis. Hence, we will write the wrenches as three-dimensional vectors, \( \xi = (f_x, f_y, m_z) \).

Thus, the reciprocal product of a wrench \( \xi = (f, m) \) and a twist \( \xi = (\omega, \nu_z) \) is given by:

\[
\mathbf{N} \mathbf{r} = \mathbf{m} \omega + \mathbf{f} \nu_z = (f_x, f_y) \omega + f_z \nu_z.
\]

### 3 Instantaneous Kinematics of PPMs

We will use the characters \( O, A, B \) as superscripts to label the three joints in the \( i \)-th leg \((i = 1, 2, 3) \) starting from the base. When a joint is revolute, its character index, with a subscript \( i \), will also be used to denote the center point of that joint, \( O_i, A_i, \) or \( B_i \) (see Figs. 3, 5, 6, 8, 9, and 11). In addition, let \( Ox \) and \( Cy \) be the base and platform frames, respectively.

#### 3.1 The Input-Output Velocity Equation

The relationship between the instantaneous motion of the platform, the output twist \( \xi = (\omega, \nu_z) \), and the nine joint rates \( \dot{\theta}_j \) \((J = O, A, B; i = 1, 2, 3) \) is given by the twist equations of the legs:

\[
\xi = \dot{\theta}_1 \mathbf{S}_1 + \dot{\theta}_2 \mathbf{S}_2 + \dot{\theta}_3 \mathbf{S}_3, \quad i = 1, 2, 3.
\]

Equation (2) is a necessary and sufficient condition for the twist \( \xi \) and the joint velocities \( \dot{\theta}_j \) about the joint screws \( \xi_j \) to be feasible at a given configuration.

To eliminate the passive joint velocities from Eq. (2) and obtain an input-output velocity equation each of the three twist equations in Eq. (2) is multiplied (via the reciprocal screw product) with a screw, \( \xi, \) reciprocal to all passive joint twists in the \( i \)-th leg. This is a wrench which, if applied to the platform, can be resisted using only the actuator of the leg. As a result, three scalar equations are obtained:

\[
\xi_i \xi_j = \xi_i \xi_j \dot{\theta}_j = \delta_{ij}, \quad i, j = 1, 2, 3,
\]

where the superscript \( a \in \{O, A, B\} \) denotes the active joint.

Equations (3) are equivalent to Eqs. (2) only if each three-dimensional reciprocal wrench, \( \xi, \) is unique, i.e., if there are no other wrenches reciprocal to both passive-joint screws in the \( i \)-th leg \[9\]. This is so if, and only if, the two passive-joint screws for each leg are linearly independent. If they are linearly dependent, there will be two reciprocal wrenches, \( \xi' \) and \( \xi'', \) and the \( i \)-th input-output velocity relationship in Eqs. (3) will need to be replaced by two equations:

\[
\xi_i' \xi_j = \xi_i' \xi_j \dot{\theta}_j = \delta_{ij},
\]

\[
\xi_i'' \xi_j = \xi_i'' \xi_j \dot{\theta}_j = \delta_{ij}.
\]

which the input and output velocities must satisfy in addition to Eqs. (3) for the remaining values of \( i \). An accurate velocity and singularity analysis must take Eqs. (4) into account.

Thus, for each of the PPMs, we expect to derive an input-output velocity equation of the type

\[
\begin{bmatrix}
\xi_{i1} \\
\xi_{i2} \\
\xi_{i3}
\end{bmatrix} =
\begin{bmatrix}
\omega \\
\nu_z
\end{bmatrix}
= \begin{bmatrix}
\xi_{i1} \xi_{i1}' & 0 & 0 \\
0 & \xi_{i2} \xi_{i2}' & 0 \\
0 & 0 & \xi_{i3} \xi_{i3}'
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix},
\]

where \( \xi \) is a shorthand for the row vector obtained from \( \xi \) by writing the moment before the force coordinates, \( \xi = [m, f_x, f_y] \).

For some PPMs there will be configurations where one or more of the equations will need to be replaced by pairs of equations like Eqs. (4) in order to accurately describe the relationships between the input and output velocities.

We will denote the matrices multiplying the platform twist and the active joint rates with \( \mathbf{Z} \) and \( \mathbf{A} \), respectively. Thus, for every configuration of every PPM, there is an equation

\[
\mathbf{Z} \xi = \mathbf{A} \dot{\mathbf{B}}.
\]

which completely describes the velocity kinematics of the mechanism. These two matrices are usually referred to as Jacobian matrices although this is not correct in the strictest mathematical sense. The matrices are \( 3 \times 3 \) almost everywhere in the configuration space, but for some PPMs there are configurations where these matrices become rectangular.

#### 3.2 Possible Reciprocal Screws for PPMs

Let us consider the nature of the reciprocal screws \( \xi \) for the different PPMs. The screw \( \xi \) will depend on the two passive joint screws in the \( i \)-th leg. We will also examine in what cases there can be configurations where the reciprocal screws for the leg form a 2-system, \( \text{Span}(\xi, \xi') \).

If the two passive joints are revolute, the reciprocal screw is a zero-pitch screw (i.e., a pure force) with an axis lying in the \( xy \) plane and intersecting the centres of the two passive \( R \) joints (Fig. 2). The two \( R \)-joint screws can become linearly dependent only when the passive joints are the extremal joints of the leg (i.e., joint \( A \) is active) and the constant link parameters have special values allowing the two joint centres to coincide. More precisely, in an \( RR \) leg this will mean that the distal and proximal links must have equal lengths \((L_1 = L_2)\), while for an \( RPR \) leg a zero offset between the two \( R \) joints will be needed \((\ell = 0)\). If the passive joint centres do coincide, there is a 2-system of reciprocal screws, namely a planar pencil of zero-pitch screws lying in the \( xy \) plane.

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Table 1: All possible 3-DOF planar legs

|  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|
| RRR | RPR | RPP | PRR | PPR | RRP | PRR | PPR | RRP | RRP |
and passing through the common center of the coinciding joints. As a result, the leg will generate two scalar input-output equations, and the two reciprocal screws \( \xi \) and \( \xi’ \) can be taken as any two different forces through the common joint center.

When one of the passive joints is revolute and the other is prismatic, they will always remain linearly independent and the uniquely defined reciprocal screw is a pure force passing through the R-joint center and perpendicular to the direction of the P joint (Fig. 2).

Finally, let us consider the possibility of two passive prismatic joints. The active remaining joint must be revolute since a planar PPP leg would have only 2 DOFs and cannot be used as a leg of a 3-DOF PPM. Let us assume that the leg is nonsingular and hence the two passive-joint translations are distinct. Then, the unique reciprocal screw is the pure moment about the z axis, i.e., \( \tau_z \). Note that this wrench does not depend on the actual configuration or the link parameters. Therefore, if all three legs are with two passive P joints, then the three reciprocal screws \( \xi \) will be identical. This means that the left-hand sides of the three Eqs. (3) will be identical and all three rows of the matrix multiplying the output twist in Eq. (5) will be \([1, 0, 0]\). Hence, the linear velocity of the platform can be arbitrary and cannot be controlled by the actuators. If only two legs have a pair of passive P joints there will still be a pair of equations which will not contain linear velocity terms and therefore there will be an uncontrollable translation of the end-effector. We can conclude that PPMs with two passive prismatic joints in each of at least two legs cannot provide controlled 3-DOF planar motion.

However, a PPM can have a single leg with two passive P joints and one active R joint, since the linear motion will be controlled by the other two legs. If the leg is PRP, it can become singular when the translations in the P joints become parallel. Then, there will be a 2-system which, apart from \( \tau_z \), includes all pure forces with axes perpendicular to the common direction of the P joints. A brief discussion of mechanisms with one RPP, PRP, or PPR leg appears in Section 5.2.

Since our main attention is directed to PPMs with identical joint sequences in all legs, we can conclude that at least one of the passive joints in a leg is revolute and therefore that the reciprocal screws \( \xi \) are all pure forces in the plane of motion as shown in Fig. 2.

3.3 Obtaining the Velocity Equation for Each PPM

When analysing a specific PPM one needs to find the reciprocal screws \( \xi \) as expressions of the chosen joint and link parameters and substitute them in Eq. (6). Matrix \( Z \) will depend on the choice of the reference frame. However, \( A \) will be coordinate invariant due to the invariance of the reciprocal product.

Since all reciprocal screws are pure forces, their coordinates will be of the type \((f_i, m_i)\), where \( f_i = (f_{xi}, f_{yi}) \) is a unit vector parallel to the force and \( m_i \) is the moment of the force axis about the platform center. These quantities will be expressed as functions of the chosen parameters. Namely, in this paper, we will use as parameters the position \((x, y)\) of the platform center \( C \), and the orientation of the platform \( \phi \) which will be assumed to be constant for the description of the singularity loci. These functions will be continuous everywhere except in those special configurations, where two passive joint screws become linearly dependent. For those configurations, the input-output velocity equation needs to be defined separately with four (or more) equations rather than three. The two reciprocal wrenches corresponding to the leg with coinciding passive revolute joints can be taken as the two forces parallel to the \( x \) and \( y \) axes and passing through the coinciding joint center \( O_2 \), \((x_{OO_2}, y_{OO_2})\):

\[
\xi = (1,0,-y_{CB}),
\]

\[
\xi' = (0,1,x_{CB}),
\]

where \( x_{CB} \) and \( y_{CB} \) are the coordinates of the platform revolute joints in an instantaneous frame with origin at the platform frame center and axes parallel to those of the base frame.

The reciprocal products on the diagonal of \( A \) will be:

\[
\xi \cdot \xi' = \lambda_i = \left( \begin{array}{c} (r_i^a \times f_i)_z \text{ if joint } a \text{ is revolute} \\ f_i \cdot v_i^a \text{ if joint } a \text{ is prismatic} \end{array} \right)
\]

where \( r_i^a \) is any vector originating at the center of the active \( R \) joint and ending on the axis of \( \xi \), and \( v_i^a \) is the unit vector defining the direction of the active \( P \) joint. When the active joint is revolute, the scalar \( \lambda_i \) is the moment of the reciprocal force with respect to the center of the active \( R \) joint. When the actuator is prismatic, \( \lambda_i \) is the projection of the force onto the direction of the actuated translation. The expression \((\cdot)\), stands for the \( z \) component of the vector argument. Thus, to write the input-output Eq. (6) for any configuration we need to express the quantities \( f_{xi}, f_{yi}, m_i \), and \( \lambda_i \) as functions of \( x \) and \( y \) and the design parameters.

3.4 Types of Singularities

Equation (6), where the matrices \( Z \) and \( A \) are occasionally non-square, completely describes the instantaneous kinematics of a parallel manipulator and hence can be used to fully describe and classify the singular configurations of the mechanism [9].

An input-output velocity equation was first used for the purposes of singularity classification of parallel mechanisms in [13]. In that paper, two main singularity types were defined. The first, or Type 1, occurs when matrix \( A \) is singular, while Type 2 corresponds to configurations where \( Z \) is singular.

In [14] a more detailed classification (for arbitrary mechanisms), based on six singularity types, was introduced. The six types are: Redundant/Impossible Input/Output (RI, II, RO, IO); Redundant Passive Motion (RPM); and Increased Instantaneous Mobility (IIM). As it was shown in [10] (Theorems 2 and 3), the two basic types described in [13] are, in fact, the types RI and RO. Namely, \( Z \) is rank deficient (Type 2) if, and only if, there is Redundant Output, i.e., an uncontrollable motion of the platform when the actuators are locked; and matrix \( A \) is rank deficient (Type 1) if, and only if, there is Redundant Input, i.e., a non-zero instantaneous motion in the active joints when the platform is fixed.

If we consider only “usual” configurations, where the matrices \( Z \) and \( A \) are square, Type 1 singularities are identical with Impossible Output (IO) singularities, i.e., configurations where the platform loses a degree of freedom because one of the legs is singular (the three joint screws in the leg are linearly dependent).

In the special configurations where \( Z \) and \( A \) are not square, a leg is always singular and IO (reduced freedom of the platform) is always present. However, this is not necessarily accompanied by the rank deficiency of the rectangular matrix \( A \) or (equivalently) by a Redundant Input motion. Instead, in such configurations, there always exists a Redundant Passive Motion (RPM), i.e., a motion of the mechanism involving only the passive joints and leaving the platform fixed. Moreover, RPM singularities will usually be Impossible Input (II) singularities as well, which means that the input velocities cannot be chosen independently.
For all configurations, a Type 2 (or RO) singularity, where an uncontrollable motion of the platform occurs, is present if, and only if, the reciprocal screws \( \xi_i \), possibly including \( \xi_5 \) and \( \xi_6 \) for some \( i \), span at most a two-system rather than the whole wrench space \( W = \text{Span}(\xi_1, \xi_2, \tau) \).

Any redundant motion (RO, RI, or RPM) can be either infinitesimal or finite. In the latter case, it is said that there is a (finite) self motion of the manipulator [15]. When, the singularity is of Type 2 (RO), it is usually referred to as an architecture singularity [16]. In general, whether the motion is infinitesimal or finite cannot be detected by studying only Eq. (6) for a single given configuration.

4 Singularity Analysis of PPMs

This section presents the principal part of our paper, namely, the results from the singularity analysis of all ten PPMs. These results were obtained using the powerful tools of planar screw theory described in the last two sections. Two are the main contributions of the present section.

Firstly, we derive the expressions for \( f_i \), \( m_i \), and \( \lambda_i \), as functions of \( x, y \), and the design parameters. This task is not always trivial but due to length limitations we will only present the final expressions. Their derivation, in essence, amounts to solving the IKP. However, we do not always need to explicitly obtain the IKP solution. In fact, for some PPMs with active \( P \) joints, we will not even define the active joint variables.

Our second contribution is the study of the singularities for each PPM. The singularity loci of Type 1 and Type 2 are determined for a constant orientation of the mobile platform. Note that the determination of the minimal-degree polynomial corresponding to Type 2 singularity loci for some PPMs is a delicate task. Those determinations cannot be performed simply by using a brute force approach with a computer algebra system such as Maple. The most difficult derivation is for 3-RRR PPMs and was already presented in a conference paper [55]. Again, due to space limitations, we will include only three figures showing the singularity loci of three PPMs (Figs. 4, 7 and 10). The singularity loci of Type 2 were obtained by a discretisation method. For a more detailed discussion on the singularity analysis of PPMs, refer to the first author’s dissertation [17].

4.1 Singularity Analysis of 3-RPR PPMs

Referring to Fig. 3, we denote with \( O_i \) and \( B_i \) the centers of the base and platform \( R \) joints, respectively. Point \( A_i \) is at the intersection of a line through \( O_i \) with the direction of the \( P \) joint, and a line through \( B_i \) and perpendicular to the first line. The length of \( O_i A_i \) is \( \rho_i^P \), which is the active joint variable, and the length of \( A_i B_i \) is \( \ell \), which is referred to as the offset. As mentioned before, the components of the vectors \( O_i A_i \) (which are constant) and \( C B_i \), (which are constant for a constant orientation) in the base frame are \((x_{AO_i}, y_{AO_i})\) and \((x_{CB_i}, y_{CB_i})\), respectively. Without loss of generality, we set \( x_{AO_i} = y_{AO_i} = x_{CB_i} = y_{CB_i} = 0 \).

If \( O_i \neq B_i \), the reciprocal screws \( \xi_i^P \), \( i = 1, 2, 3 \), are the screws passing through the two \( R \) joints in each leg:

\[
f_i = \frac{1}{v_i} [\vec{x}_i, \vec{y}_i],
\]

\[
m_i = (CB_i \times f)_i = \left[ \frac{-y_{CB_i}}{x_{CB_i}} \right],
\]

where

\[
\vec{x}_i = x - x_{AO_i} + x_{CB_i}, \quad \vec{y}_i = y - y_{AO_i} + y_{CB_i},
\]

\[
v_i = \vec{x}_i, \quad \vec{y}_i.
\]

Note that \((\vec{x}_i, \vec{y}_i)\) are the components of vector \( \overrightarrow{O_i B_i} \) while \( v_i \) is its length. If \( O_i = B_i \), the diagonal elements of \( A \) are

\[
\lambda_i = \rho_i^P v_i, \quad i = 1, 2, 3,
\]

where

\[
\rho_i^P = \frac{1}{\delta_i} \sqrt{\vec{x}_i^2 + \vec{y}_i^2 - \ell^2}, \quad \delta_i = \pm 1.
\]

Equation (14) provides the solution of the inverse kinematic problem (IKP), being real only within vertex space \( i \) which is the exterior of a circle centred at \((x_{AO_i} - x_{CB_i}, y_{AO_i} - y_{CB_i})\) and of radius \( \delta \). Vertex space \( i \) is defined as the area where the platform center, \( C \), is constrained to lie taking into account the kinematic constraint imposed by leg \( i \) and the constant orientation of the platform. Hence, the form of each vertex space is fixed for a given PPM, while its position varies as a function of the platform orientation. The constant-orientation workspace is the intersection of all three vertex spaces.

Type 2 singularities occur when the three screw axes intersect or are parallel, i.e., when \( \det(Z) = 0 \). This determinant consists of a fraction whose numerator is a quadratic, while the denominator is \( v_1^2 v_2 v_3 \). Note that the quadratic does not depend on \( \ell \). The properties of the corresponding quadratic curve as functions of the design parameters have been already studied in detail [4]. We would only add the fact that the curve always passes through the three vertex space centers. For some designs though, with congruent base and platform, there is an orientation at which the PPM is singular for any position. In general, though, the Type 2 singularity loci are those parts of the quadratic curve that are within all vertex spaces.

Two different cases may occur when \( \rho_i^P = 0 \) depending on the value of the offset \( \ell \). If \( \ell = 0 \), i.e., \( O_i = B_i \), there are two linearly independent reciprocal screws for the degenerate leg and Eq. (9) is not valid. This singularity allows an uncontrollable passive motion (RPM type) and the three input velocities cannot be chosen independently (II type) [9]. This redundant motion is full-cycle rather than infinitesimal. Note that, although there is a loss of a degree of freedom of the mobile platform (IO type), strictly speaking, this is not a Type 1 singularity since \( A \) is not singular. If \( \ell \neq 0 \), the singularity loci of Type 1 are simply the vertex space boundaries (circles).

4.2 Singularity Analysis of 3-RPR PPMs

Referring again to Fig. 3, the active joint variables \( \vec{x}_i^P \) for this PPM are the angles \( \theta_i^P \). The reciprocal screws \( \xi_i^P \), \( i = 1, 2, 3 \), are the screws passing through the platform \( R \) joints and normal to the direction of the corresponding \( P \) joints:

\[
f_i = \frac{-\sin \theta_i^P}{\cos \theta_i^P} \left[ \frac{\ell \vec{x}_i - \rho_i^P \vec{y}_i}{\sqrt{\vec{x}_i^2 + \vec{y}_i^2}}, \frac{\ell \vec{y}_i + \rho_i^P \vec{x}_i}{\sqrt{\vec{x}_i^2 + \vec{y}_i^2}} \right].
\]
The parameters $\bar{x}$, $\bar{y}$, $\ell$, $\theta_i^0$, and $p_i^0$ as well as the moment $m_i$ are as defined in Section 4.1. If $O_i \neq B_i$, only the first part of Eq. (15) remains valid. The diagonal elements of $\Lambda$ are

$$
\lambda_i = \rho_i^0, \quad i = 1, 2, 3.
$$

(16)

Type 1 singularities occur for the same configurations as in 3-RPR PPMs. For this mechanism, however, when $p_i^0 = 0$ and $\ell = 0$ (i.e., $O_i = B_i$), there is a generic Type 1 (RI) singularity, where the input velocities are indeterminate. This redundant input (RI) motion is full-cycle.

When $\ell = 0$, the determinant of $A$ consists of a fraction whose numerator is again a quadratic polynomial and whose denominator is $p_i^0 \rho_i^0$. Again, the corresponding quadratic curve defining Type 2 singularities always passes through the centers of the vertex spaces. Those three centers, however, do not necessarily correspond to Type 2 singularities.

This is where we need to introduce the concept of branch index, $\delta_i$, as defined in Eq. (14). As seen from that equation, RPR legs have two solutions (branches) to their IKP. Hence, 3-RPR and 3-RPM PPMs have eight solutions to their IKP, called working modes [18]. While the Type 2 singularity loci for both manipulators in the case $\ell = 0$ are the same for all working modes, they differ for 3-RPR PPMs with $\ell \neq 0$.

When $\ell \neq 0$, the Type 2 singularities for 3-RPR PPMs are completely different from the case $\ell = 0$. The numerator of $\det(A)$ is no longer a polynomial but contains radicals (the variables $p_i^0$), and is, hence, dependent on the given working mode. If we manipulate properly that expression and raise it to square three times, we may obtain a polynomial of degree 16 in $x$ and $y$. This polynomial will cover all working modes.

An example of the singularity loci for a 3-RPR PPM for all working modes is presented in Fig. 4 ($\ell = 50$, $x_{O01} = y_{O01} = x_{CB1} = y_{CB1} = 0$, $x_{OO1} = -255.6$, $y_{OO1} = 0$, $x_{CB2} = -110.8$, $y_{CB2} = 0$, $x_{OO2} = -127.8$, $y_{OO2} = -221.4$, $x_{CB3} = -55.4$, $y_{CB3} = -95.9$). In that figure, as well as in Figs. 7 and 10, the boundaries of the constant-orientation workspace, i.e., the Type 1 singularity loci, are drawn in dashed line. The Type 2 singularity loci are drawn in solid line. Each of the points of contact between the two types of singularity loci correspond to a change of working mode.

Type 2 singularity loci of PPMs with reciprocal screws dependent on the chosen working mode are always within the union of the three vertex spaces and even, usually, within their intersection, i.e., within the constant-orientation workspace.

4.3 Singularity Analysis of 3-RRR PPMs. If $O_i \neq B_i$, the reciprocal screws $\xi_i^0$, $i = 1, 2, 3$, for this architecture are the screws through the passive R joints in a leg (Fig. 5):

$$
\xi_i^0 = \frac{1}{\ell_2} \begin{bmatrix} x_i - \ell_1 \cos \theta_i^0 \\ y_i - \ell_1 \sin \theta_i^0 \end{bmatrix},
$$

(17)

where $x_i$ and $y_i$ are as in Eq. (11), while $m_i$ is as defined in Eq. (10). The diagonal elements of $\Lambda$ are

$$
\lambda_i = \frac{\ell_1 \ell_2}{\sqrt{x_i^2 + y_i^2}} \sin \theta_i^0
$$

(18)

where

$$
\Gamma_i = x_i^2 + y_i^2 - \ell_1^2 - \ell_2^2,
$$

(19)

$$
p_i = \frac{x_i^2 + y_i^2}{2 \ell_1}.
$$

(20)

A comparison between Eq. (17) and Eq. (9) reveals that Type 2 singularity loci for 3-RRR PPMs are the same as for 3-RPR PPMs. Type 1 singularities occur when the proximal ($O_iA_i$) and distal ($A_iB_i$) link in a leg are aligned. Therefore, the Type 1 singularity loci for a constant orientation are pairs of concentric circles of radius $|\ell_i \pm \ell_2|$, centered at $(x_{OO} - x_{BC}, y_{OO} - y_{BC})$. The area between a pair of circles is vertex space $i$ and is defined by the inequality $\Gamma_i \geq 0$.

When the links are of equal lengths ($\ell_1 = \ell_2$) and are overlapping, i.e., $O_i = B_i$, there is a singularity of the same class (RPM, II, IO) as in 3-RPR PPMs and not a Type 1 singularity.

Note that a 3-RRR PPM has 8 working modes, but its singularities do not depend on the particular working mode unlike a 3-RPR PPM, as we will see next.

4.4 Singularity Analysis of 3-RRR PPMs. The reciprocal screws $\xi_i^0$, $i = 1, 2, 3$, for 3-RRR PPMs are along the distal links (Fig. 5):

$$
\xi_i^0 = \frac{1}{\ell_2} \begin{bmatrix} x_i - \ell_1 \cos \theta_i^0 \\ y_i - \ell_1 \sin \theta_i^0 \end{bmatrix},
$$

(21)

where $x_i$ and $y_i$ are as defined in Eq. (11), while $m_i$ is as defined in Eq. (10). In addition:

$$
\sin \theta_i^0 = \frac{p_i x_i - y_i \delta_i \Gamma_i}{x_i^2 + y_i^2}, \quad \cos \theta_i^0 = \frac{p_i x_i - y_i \delta_i \Gamma_i}{x_i^2 + y_i^2},
$$

(22)

where $\Gamma_i$ and $p_i$ are as defined in Eqs. (19) and (20), respectively. The diagonal elements of $\Lambda$ are...
Type 1 singularity loci are the same as for 3-RRR PPMs, but when \( \ell_1 = \ell_2 \), the singularity loci include the vertex space centres as well. A recent detailed study of their Type 2 singularity loci revealed that they are represented by curves of degree 42. Note that once again Type 2 singularity loci for a constant orientation for a single working mode are represented by a non-polynomial and the polynomial of degree 42 corresponds to the singularities for all 8 working modes.

An interesting simplified design of a 3-RRR PPMs is obtained when two of the platform R joints coincide. The Type 2 singularity loci of such a mechanism reduce to four circles and one sextic and can be determined geometrically.

4.5 Singularity Analysis of 3-PRR PPMs. The 3-PRR architecture was proposed in [19]. Let in each leg, the directed line through the intermediate R joint and along the active translation be defined by its moment \( \mu_i \) about the base frame center, and its angle \( \alpha_i \) measured from the x axis. The reciprocal screws \( \xi_i^0, i = 1,2,3 \), are along the distal links (Fig. 6): \[
\xi_i^0 = \ell_i \left[ \begin{array}{c} -\sin \alpha_i \mu_i + \cos \alpha_i \delta_i \sqrt{\Gamma_i} \\ \cos \alpha_i \mu_i + \sin \alpha_i \delta_i \sqrt{\Gamma_i} \end{array} \right],
\]
(24)

while \( m_i \) is as defined in Eq. (10). In Eq. (24),
\[
\Gamma_i = \ell_i^2 - p_i^2,
\]
(25)

\[
p_i = \mu_i (x + x_{CB}) \sin \alpha_i + (y + y_{CB}) \cos \alpha_i.
\]
(26)

Note that \( p_i \) is the distance from \( B_i \) to the above-mentioned directed line \( i \). Finally, the diagonal elements of \( \Lambda \) are
\[
\lambda_i = \cos \theta_i^0 = \frac{1}{\ell_i} \delta_i \sqrt{\Gamma_i}, \quad i = 1,2,3.
\]
(27)

The inequality \( \Gamma_i > 0 \) defines each vertex space which is the area between a pair of lines parallel to the direction of \( P \) joint \( i \) and separated by a distance of \( 2\ell \). Indeed, Type 1 singularities occur when a distal link is normal to the \( P \) joint in a leg. The Type 2 singularity loci for a constant orientation of the platform form a curve of degree 20 that corresponds to all working modes. An example of the singularity loci for a 3-PRR PPM for all working modes is presented in Fig. 7 (\( \ell = 130 \), \( \alpha_1 = 0 \), \( \alpha_2 = 2\pi/3 \), \( \alpha_3 = -2\pi/3 \), \( \mu_1 = \mu_3 = 0 \), \( \mu_2 = -147.6 \), \( x_{CB_1} = y_{CB_1} = y_{CB_2} = 0 \), \( x_{CB_2} = -110.8 \), \( x_{CB_1} = -55.4 \), \( y_{CB_1} = -95.9 \)). The simplified design is obtained when two platform \( R \) joints coincide, which leads to singularity loci represented by two parallel line segments, and, generally, one (portion of an) ellipse and one (arc of a) circle.

4.6 Singularity Analysis of 3-PRP PPMs. The reciprocal screws \( \xi_i^p, i = 1,2,3 \), for this architecture are the screws passing through the platform \( R \) joints and normal to the directions of the corresponding \( P \) joints:
\[
f_i = \left[ \begin{array}{c} -\sin \alpha_i \\ \cos \alpha_i \end{array} \right],
\]
(28)

where \( m_i \) is as defined in Eq. (10), and \( \alpha_i \) is the angle between the \( x \) axis and the direction of prismatic joint \( i \). As for the diagonal elements of \( \Lambda \), they are given as
\[
\lambda_i = \ell \cos \theta_i^0, \quad i = 1,2,3.
\]
(29)

Clearly, for a constant orientation of the mobile platform, the configuration of the three screws is fixed, and hence the Type 2 singularities do not depend on the platform position. It can be easily shown that there are only two pairs of opposite orientations, at which the PPM is in a singularity of Type 2 for any position but does not, generally, undergo a self motion. For any other orientation, there are no Type 2 singularities. Similarly to 3-RRR PPMs, the choice of working mode does not influence the singularities. The Type 1 singularities are the same as for 3-PRR PPMs, i.e., three pairs of parallel lines.

4.7 Singularity Analysis of 3-PPP and 3-RPP PPMs. The kinematic properties of both architectures are the same. The reciprocal screw \( i (i = 1,2,3) \) is the screw passing through the \( R \) joint and normal to the passive \( P \) joint (Fig. 8):
\[
f_i = \left[ \begin{array}{c} -\sin \gamma_i \\ \cos \gamma_i \end{array} \right],
\]
(30)

\[
m_i = -(x - x_{OO}) \cos \alpha_i - (y - y_{OO}) \sin \alpha_i.
\]
(31)

where \( \alpha_i \) is the angle between the \( x \) axis and the direction of the passive prismatic joint \( i \). The diagonal elements of \( \Lambda \) are
\[
\lambda_i = \left\{ \begin{array}{ll}
\sin \gamma_i, & \text{for } 3-PPP \text{ PPMs} \\
-\sin \gamma_i, & \text{for } 3-RPP \text{ PPMs}
\end{array} \right. \quad i = 1,2,3.
\]
(32)

where \( \gamma_i \) is the angle between the prismatic joints in leg \( i \).

Since the directions of the passive joints are defined by the platform orientation, Type 2 singularities do not depend on the platform position. It is possible to show that there are only two pairs of opposite orientations at which the PPM is in a Type 2

**Fig. 6** A 3-DOF PPM of type 3-PR or 3-PRR

\[
\lambda_i = -\ell_i \sin \theta_i^0 = -\ell_i \delta_i \sqrt{\Gamma_i}, \quad i = 1,2,3.
\]
(23)
singularity. As seen from Eq. (32), there are no Type 1 singularities (provided that no two P joints in a leg are parallel).

4.8 Singularity Analysis of 3-RRP PPMs. Let in each leg, the directed line through the R joint at A, and along the active translation be defined by its moment μi about the platform center, C, and its angle αi measured from the base x axis. The reciprocal screws ζi, i=1,2,3, for this architecture are the screws passing through the passive R joints and normal to the corresponding P joints (Fig. 9):

\[
f_i = \begin{bmatrix} -\sin \alpha_i \\ \cos \alpha_i \end{bmatrix}, \quad \text{ gim } = (x-x_oo) \cos \alpha_i - (y-y_oo) \sin \alpha_i + \delta_i \Gamma_i, \quad \text{ (33)}
\]

where

\[
\Gamma_i = \ell^2 - (\mu_i + (x-x_oo)) \sin \alpha_i - (y-y_oo) \cos \alpha_i. \quad \text{ (34)}
\]

The diagonal elements of A are

\[
\lambda_i = \delta_i \sqrt{\Gamma_i}, \quad i=1,2,3. \quad \text{ (36)}
\]

This is another PPM with 8 working modes that all lead to different singularities. The Type 2 singularity loci for all working modes are represented by a polynomial of degree 6. Type 1 singularities occur when a proximal link is normal to the direction of the corresponding P joint. Indeed, each vertex space is the area between a pair of parallel lines as in the case of 3-PRR and 3-PRR PPMs. An example of the singularity loci for a 3-RPR PPM for all working modes is presented in Fig. 10 (\(i = 130\), \(\alpha_1 = 0\), \(\alpha_2 = -2\pi/3\), \(\alpha_3 = 2\pi/3\), \(\mu_1 = \mu_2 = \mu_3 = 0\), \(x_oo = y_oo = 0\), \(x_oo = -66.4\), \(y_oo = -66.4\)).

An interesting simplified design may be obtained by constructing two of the P joints to be parallel, while setting the third one, e.g., normal to them. In that case, it is easy to see that a Type 2 singular configuration will only occur when the passive R joints of the first two legs coincide. In fact, the same principle for design simplifications may be applied to 3-RPP, 3-RPP, and 3-PRR PPMs.

4.9 Singularity Analysis of 3-PRP PPMs. A design of this type was proposed in [20] under the name double-triangular manipulator. In addition, a 3-PRP alignment stage, based on the just previously mentioned singularity-free design is commercially available [21].

Let in each leg, the directed line through the R joint and along the active (passive) translation be defined by its moment \(\mu_i^O\) (\(\mu_i^P\)) about the base (platform) frame center, O (C), and its angle \(\alpha_i^O\) (\(\alpha_i^P\)) measured from the base x axis. The reciprocal screws, \(\zeta_i\), i=1,2,3, are the screws passing through the R joints and normal to the passive P joints (Fig. 11):

\[
f_i = \begin{bmatrix} -\sin \alpha_i^P \\ \cos \alpha_i^P \end{bmatrix}, \quad \text{ gim } = \mu_i^O - x \sin \alpha_i^O + y \cos \alpha_i^O - \cos (\alpha_i^O - \alpha_i^P) \mu_i^P, \quad \text{ (37)}
\]

As for the diagonal elements of A, they are

\[
\lambda_i = \sin(\alpha_i^O - \alpha_i^P), \quad i=1,2,3. \quad \text{ (39)}
\]

Type 1 singularities occur when the directions of both P joints in a leg coincide. Hence, these singularities occur only at three pairs of opposite orientations, and at each such orientation, the Type 1 singularity loci form a line. This refutes the common misconception that Type 1 singularities are always the workspace boundaries.
In general, the Type 2 singularity loci for a constant orientation are also represented by a line. However, for designs for which \( \alpha_i^0 = \alpha_i^0 + \phi \) (where \( i = 1, 2, 3 \)), there are only two orientations at which the PPM is at a Type 2 singularity, and that, for any position of the mobile platform. Moreover, for most designs, a pose that is in a Type 1 singularity can be made to be in a Type 2 singularity as well, by adjusting the corresponding \( R \) joint. The only design of a 3-\( PRP \) PPM that is completely free of Type 2 singularities is that for which two, and only two, of the passive \( P \) joints are parallel.

5 Further Remarks

5.1 Parameterization and Polynomial Derivation. In this paper, we use the minimal and most intuitive parameterization for all ten architectures. In the case of base and/or platform \( R \) joints, we use the coordinates of vectors \( \overrightarrow{OO_i} \) and \( \overrightarrow{CB_i} \) expressed in the base frame, \((x_{OO_i}, y_{OO_i})\) and \((x_{CB_i}, y_{CB_i})\), respectively. In the case of base and/or platform \( P \) joints, we use the angle between the base \( x \) axis and the direction of the \( P \) joint, \( \alpha_i \), and the moment, \( \mu_1 \), of the line directed along the \( P \) joint and through the neighboring \( R \) joint about the base and platform centers, respectively. In addition, we assume without loss of generality that some of the parameters are zeros.

The described parameterization is very useful for the elegant derivation of the matrices \( Z \) and \( \Lambda \). In most cases in which the active joints are prismatic we do not even have to define the active joint variables. Consequently, we do not need to present the solution to the IKP.

However, a different parameterization may be more convenient for the derivation of the polynomial representing the Type 2 singularity loci of a PPM with multiple working modes. Namely, instead of using \( x_{OO_i} \) and \( y_{OO_i} \) as parameters, \( x_{CB_i} - x_{OO_i} \) and \( y_{CB_i} - y_{OO_i} \) should be used [5]. The latter is essential in order to reduce the number of terms.

5.2 PPMs with Mixed Legs. Firstly, let us consider the case of a PPM with any combination of three legs from the ten studied previously. For such a PPM, one should simply write the scalar velocity equation for each leg as given in this paper and then combine all three to obtain the matrices \( Z \) and \( \Lambda \). Note that one can easily reverse the order of the joints (e.g., use an \( RRR \) chain and an \( RPR \) one) and still use our equations. To obtain the Type 2 singularity loci, one should simply compute \( \det(Z) \) in order to obtain a polynomial in \( x \) and \( y \).

Secondly, let us consider the case of a PPM with a leg having two passive \( P \) joints and one active \( R \) joint (\( RPP \), \( PPR \), \( PRP \)) and any combination of two legs from the ten different studied previously. As we already mentioned in Section 3.2, the reciprocal screw of the leg with the two passive \( P \) joints is the moment about the \( z \) axis. Hence, for the PPM to be in a Type 2 singularity, the reciprocal screws of the two other legs, which are forces in the \( xy \) plane, should be linearly dependent. Thus, we only need to obtain the expressions for \( f_x \) and \( f_y \) and check when their vector product vanishes.

In both cases, the Type 1 singularity loci can be obtained directly from the diagonal elements of \( A \). However, the loci may also determined geometrically in a much easier way. For each leg, the Type 1 singularity loci, if there are any, are either a (pair of parallel) line(s) or a (pair of concentric) circle(s).

5.3 The Missing "Type 3" Singularities. Some readers might ask why we have omitted the so-called Type 3 singularities. This type was introduced in [13] in conjunction with Type 1 and Type 2 singularities. It was defined as the type of singular configurations at which matrices \( Z \) and \( \Lambda \) are both singular. Then, the type was erroneously identified with architecture singularities. It was later claimed that Type 3 singularities are only a subset of architecture singularities [16]. This inaccurate definition may even be found in a recent textbook [8, p. 226]. As we show, architecture singularities, or finite self motions, are not related to the simultaneous degeneracy of matrices \( Z \) and \( \Lambda \).

Indeed, for a parallel mechanism, the singularity loci of Type 2 generally intersect the vertex space boundary, i.e., the singularity loci of Type 1 and no special design conditions are required for that. One obvious example is 3-\( RPR \) PPMs, where, for a constant orientation, the intersection points of the Type 2 singularity loci (a quadratic curve) and Type 1 singularity loci (three circles) always exist for any design with \( \ell \neq 0 \). We have observed that configurations at which both matrices \( Z \) and \( \Lambda \) are singular exist for the general designs of most 3-DOF PPMs. However, generally, there is no finite uncontrollable motion in these configurations.

In conclusion, those special configurations \((i)\) do not occur for particular designs only and \((ii)\) do not necessarily correspond to self motions.

5.4 PPMs with Parallelograms. A parallelogram is a four-bar mechanism whose opposing links are of equal length (Fig. 12(a)). It has the property that its opposing links remain parallel at all times. The parallelogram is sometimes used in the construction of mechanisms as a 1-DOF pair (denoted as \( Pa \)) in combination with \( P \) and \( R \) joints. However, we would like to stress that the parallelogram is not a distinct kinematic pair in the strictest sense.

Indeed, an exhaustive study of 3-DOF chains based on one passive/active \( Pa \) joint and any combination of \( P \) or \( R \) joints, placed in any order, shows that any such chain has a kinematically equivalent 3-DOF serial chain with only three \( P \) and/or \( R \) joints.
Particularly, for any 3-DOF PPM with identical legs that have $Pa$ joints, there is a kinematically equivalent 3-DOF PPM with chains of type $RRR$, $RRR$, $PRR$, or $PRR$. By kinematically equivalent PPM, we mean a PPM that has the same input-output velocity equation, and the same singularities. For example, the 3-DOF $3$-$RPaR$ mechanism whose leg architecture is shown in Fig. 12 is kinematically equivalent to a 3-DOF $3$-$RRR$ PPM. In all cases, the $Pa$ joint is substituted by an $R$ joint, but the order of the three joints does not always remain the same (e.g., $3$-$PaRP$ PPMs are kinematically equivalent to $3$-$PRR$ PPMs). This is an interesting fact, since, instantaneously, a $Pa$ joint is equivalent to a translation.

Hence, the main reasons for using parallelograms may be to avoid link interference and to achieve static or even dynamic balancing (e.g. [22]). Of course, parallelograms are also ideal if chains with two actuators are needed, in which case, both drives can be placed at the base.

6 Conclusions

The basic results of an exhaustive study on the singularities of all types of 3-DOF planar fully-parallel mechanisms with three identical legs have been presented. The compact and systematic presentation was possible due to the use of reciprocal screws. We have presented a description of the method of reciprocal screws for planar mechanisms.

Furthermore, we have obtained the Type 2 singularity loci for all architectures, using a discretisation method, and the Type 1 loci, using a geometrical method. The Type 2 singularity loci are sometimes very complex curves. The knowledge of the shape and nature of these loci is very useful for the choice and design of an architecture for a particular task.

The studied mechanisms have only prismatic and revolute joints. It was stated that the parallelogram that is sometimes used as a 1-DOF kinematic pair can always be substituted with a revolute joint.

References